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# Boundary behavior of the Bergman kernel function on pseudoconvex domains with comparable Levi form

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## Abstract

Let  $\Omega$  be a smoothly bounded pseudoconvex domain in  $\mathbb{C}^n$  and let  $z_0 \in b\Omega$  be a point of finite type. We also assume that the Levi form of  $b\Omega$  is comparable in a neighborhood of  $z_0$ . Then we get a quantity which bounds from above and below the Bergman kernel function in a small constant and large constant sense.

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## 1. Introduction

In this paper we want to get the boundary behavior of the Bergman kernel function  $K(z, \bar{z})$ , restricted to the diagonal, for smoothly bounded pseudoconvex domains in  $\mathbb{C}^n$  near a point of finite type with comparable Levi form. The boundary behavior of  $K(z, \bar{z})$  is completely understood near the strictly pseudoconvex points [6–8]. For weakly pseudoconvex domains, however, we have to use different approaches in each different types of domains. In [2], Catlin treated completely the case of pseudoconvex domains of finite type in  $\mathbb{C}^2$ , and McNeal generalized Catlin's result to decoupled [13] and convex [14] pseudoconvex domains of finite type in  $\mathbb{C}^n$ . Also Herbort [10] got the boundary behavior

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of  $K(z, \bar{z})$  for pseudoconvex domains of homogeneous finite diagonal type, and the author treated completely for pseudoconvex domains of finite type with one degenerate Levi form [3,4]. For the case of diagonalizable Levi form, Fefferman et al. [9] also got the estimates on the kernel.

In the rest of this paper, we let  $\Omega$  be a smoothly bounded pseudoconvex domain in  $\mathbb{C}^n$  with smooth defining function  $r$ , i.e.,  $\Omega = \{z \in \mathbb{C}^n: r(z) < 0\}$ , and let  $K_\Omega(z, \bar{z})$  be the corresponding Bergman kernel function.

Let  $\lambda_1(z), \dots, \lambda_{n-1}(z)$  be the eigenvalues of the Levi form  $\partial\bar{\partial}r$  of tangential vector fields and assume that  $z_0 \in b\Omega$ . We say  $\Omega$  has comparable Levi form near  $z_0$  if there are a constant  $c > 0$  and a neighborhood  $U$  of  $z_0$  such that

$$\lambda_k(z) \geq c \sum_{i=1}^{n-1} \lambda_i(z), \quad k = 1, 2, \dots, n-1, \quad z \in U. \quad (1.1)$$

For example, let  $r(z) = 2\operatorname{Re} z_3 + (|z_1|^2 + |z_2|^2)^2$  be a defining function for a domain  $\Omega$  in  $\mathbb{C}^3$  near the origin. Then the Levi form of  $b\Omega$  satisfies (1.1) near the origin, and hence  $\Omega$  has a comparable Levi form near the origin. Recently, Koenig [12] has studied this class of domain and obtained the maximal Sobolev and Hölder estimates for the tangential Cauchy–Riemann operator and boundary Laplacian.

Let  $z_0 \in b\Omega$  be a point of finite type  $m$  in the sense of D’Angelo [5]. Assuming that  $|\partial r / \partial z_n(z_0)| \geq c_1 > 0$  in a neighborhood  $U$  of  $z_0$ , set

$$L_j = \frac{\partial}{\partial z_j} - \left( \frac{\partial r}{\partial z_n} \right)^{-1} \frac{\partial r}{\partial z_j} \frac{\partial}{\partial z_n}, \quad j = 1, 2, \dots, n-1, \quad L_n = \frac{\partial}{\partial z_n}. \quad (1.2)$$

Then  $\{L_1, \dots, L_n\}$  form a basis of  $\mathbb{C}T^{(1,0)}(U)$  provided  $U$  is sufficiently small. For any integer  $j, k > 0$ , set

$$\mathcal{L}_{j,k} \partial\bar{\partial}r(z) = \underbrace{L_1 \dots L_1}_{(j-1) \text{ times}} \underbrace{\bar{L}_1 \dots \bar{L}_1}_{(k-1) \text{ times}} \partial\bar{\partial}r(z)(L_1, \bar{L}_1)(z), \quad (1.3)$$

and define

$$C_l(z) = \max\{|\mathcal{L}_{j,k} \partial\bar{\partial}r(z)|: j+k=l\},$$

$$M(z) = \sum_{l=2}^m C_l(z)^{2(n-1)/l} |r(z)|^{-2(n-1)/l}. \quad (1.4)$$

Then we can state our main result as follows.

**Theorem 1.1.** *Let  $\Omega$  be a smoothly bounded pseudoconvex domain in  $\mathbb{C}^n$ . Let  $z_0 \in b\Omega$  be a point of finite type  $m$  and assume that the Levi form of  $b\Omega$  is comparable in a neighborhood  $U$  of  $z_0$ . Then there exists a constant  $C > 0$  such that*

$$\frac{1}{C} M(z) |r(z)|^{-2} \leq K_\Omega(z, \bar{z}) \leq C M(z) |r(z)|^{-2} \quad (1.5)$$

for all  $z \in U$ , where  $M(z)$  is defined as in (1.4).

The proof of the above theorem is based on the study of local geometry of  $b\Omega$  near  $z_0 \in b\Omega$  where the Levi form is comparable (Lemma 2.4). By virtue of condition (1.1), we can compare the Levi forms, and quantity (1.3) controls the other terms,  $\partial\bar{\partial}r(L_j, \bar{L}_k)$ ,  $1 \leq j \leq n-1$ ,  $2 \leq k \leq n-1$ , and their derivatives. Then the proof of Theorem 1 is reduced to the construction of the largest polydisc  $B(\tilde{z})$  centered at each point  $\tilde{z} \in U$ , and the construction of strictly plurisubharmonic weight function with maximal Hessian on  $B(\tilde{z})$  [2–4].

In the sequel, we let  $A \lesssim B$  (or  $A \approx B$ ) denote that there is a constant  $C$  (independent of the quantities in  $A$  or  $B$ ) such that  $A \leq CB$  (or  $A/C \leq B \leq CA$ ).

## 2. Estimates of the Bergman kernel

In this section we want to analyze the local geometry of  $b\Omega$  near a point of finite type where the Levi form is uniformly comparable. Lemma 2.4 is a crucial one for this paper and the condition of comparable eigenvalues of the Levi form is essentially used. Then the estimates of the Bergman kernel function, restricted to the diagonal, follows from the routine estimates as in [2].

Let  $\{L_1, \dots, L_n\}$  be the local frame of  $\mathbb{C}T^{(1,0)}(U)$  defined in Section 1, and set  $A = (c_{jk})_{1 \leq j, k \leq n-1}$ , where  $c_{jk} = \partial\bar{\partial}r(z)(L_j, \bar{L}_k)$ . Let  $\Lambda$  be the diagonal matrix whose diagonal entries,  $\lambda_1(z), \dots, \lambda_{n-1}(z)$ , are the eigenvalues of  $A$ . Let  $P = (p_{j,k})$  be the unitary matrix which satisfies  $P^*AP = \Lambda$ . Then we can write

$$c_{j,k} = \sum_l p_{lj} \lambda_l p_{lk}^* = \sum_l \lambda_l p_{kl} p_{lj},$$

and hence it follows from (1.1) that

$$c_{kk} = \sum_l \lambda_l |p_{kl}|^2 \approx \lambda_k \approx \lambda_1, \quad 1 \leq k \leq n-1. \quad (2.1)$$

If  $j \neq k$ ,  $1 \leq j, k \leq n-1$ , we also have

$$|c_{jk}| \lesssim \lambda_k \approx \lambda_1. \quad (2.2)$$

Since  $\lambda_j(z) \approx \lambda_k(z)$  for  $z \in U$ , there is a uniform constant  $C > 1$  such that

$$\frac{1}{C} \lambda_j(z) \leq \lambda_k(z) \leq C \lambda_j(z), \quad (2.3)$$

for all  $z \in U$ ,  $1 \leq j, k \leq n-1$ .

Let  $\alpha, \beta$  be multi-indices and let  $\alpha' = (\alpha_1, \dots, \alpha_{n-1}, 0)$ ,  $\alpha'' = (0, \alpha_2, \dots, \alpha_{n-1}, 0)$ , etc. Also let  $\partial^\alpha$  denote the holomorphic differential operator of order  $|\alpha|$ . We first construct special coordinates centered at  $z' \in U$ .

**Proposition 2.1.** *For each  $z' \in U$  and positive integer  $m$ , there is a biholomorphism  $\Phi_{z'}: \mathbb{C}^n \rightarrow \mathbb{C}^n$ ,  $\Phi_{z'}^{-1}(z') = 0$ , satisfying*

$$\begin{aligned} \rho(\zeta) := r(\Phi_{z'}(\zeta)) &= r(z') + \operatorname{Re} \zeta_n + \sum_{\substack{j+k \leq m \\ j,k \geq 1}} a_{jk}(z') \zeta_1^j \bar{\zeta}_1^k \\ &+ \sum_{\substack{|\alpha' + \beta'| \leq m \\ |\alpha'|, |\beta'| \geq 1 \\ 1 \leq |\alpha'' + \beta''| \leq m}} b_{\alpha'\beta'}(z') \zeta^{\alpha'} \bar{\zeta}^{\beta'} + \mathcal{O}(|\zeta'|^{m+1} + |\zeta| |\zeta_n|), \end{aligned} \quad (2.4)$$

where  $\zeta' = (\zeta_1, \dots, \zeta_{n-1}, 0)$ .

**Proof.** We may assume that  $z' = 0$ . After a linear change of coordinates, we can write

$$r(z) = \operatorname{Re} z_n + \mathcal{O}(|z|^2).$$

Now assume that we have defined  $\Phi^{l-1}: \mathbb{C}^n \rightarrow \mathbb{C}^n$  so that there exist numbers  $a_{jk}(z')$  for  $j, k \geq 1$ ,  $j+k < l$ , and  $b_{\alpha'\beta'}(z')$ ,  $|\alpha' + \beta'| < l$ , so that  $\rho_l(w) := r \circ \Phi^{l-1}(w)$  satisfies

$$\begin{aligned} \rho_l(w) &= r(z') + \operatorname{Re} w_n + \sum_{\substack{j+k < l \\ j,k \geq 1}} a_{jk}(z') w_1^j \bar{w}_1^k \\ &+ \sum_{\substack{|\alpha' + \beta'| < l \\ |\alpha'|, |\beta'| \geq 1 \\ 1 \leq |\alpha'' + \beta''|}} b_{\alpha'\beta'}(z') w^{\alpha'} \bar{w}^{\beta'} + \mathcal{O}(|w'|^l + |w'| |w_n|). \end{aligned} \quad (2.5)$$

Let  $\phi^l(w) = \zeta$  be defined by

$$\zeta_j = w_j, \quad j = 1, 2, \dots, n-1,$$

and

$$\zeta_n = w_n + \frac{2}{l!} \frac{\partial^l \rho_l(0)}{\partial w_1^l} w_1^l + \sum_{\substack{|\alpha'|=l \\ |\alpha''| \geq 1}} \frac{2}{\alpha'!} \partial^{\alpha'} \rho_l(0) w^{\alpha'}.$$

Set  $\Phi^l = \Phi^{l-1} \circ \phi^l$ , and set  $\rho_{l+1} = r \circ \Phi^l$ . Then  $\rho_{l+1}$  satisfies the analog of (2.5) with  $l$  replaced by  $l+1$ . If we continue up to  $l=m$ , then we will get (2.4).  $\square$

Using the special coordinates in Proposition 2.1, for each  $z' \in U \cap \Omega$ , we want to define a quantity  $\tau(z', \delta)$  in such a way that there is a biholomorphic image of maximal polydisc on which the function changes by no more than some prescribed small number  $\delta > 0$ . Set

$$\begin{aligned} A_{l_1}(z') &= \max\{|a_{jk}(z')|: j+k=l_1\}, \\ A_{l_2}(z') &= \max\{|b_{\alpha'\beta'}(z')|: |\alpha' + \beta'|=l_2\}, \quad 2 \leq l_1, l_2 \leq m. \end{aligned}$$

For each  $\delta > 0$ , we define  $\tau(z', \delta)$  by

$$\tau(z', \delta) = \min\{(\delta/A_{l_i}(z'))^{1/l_i}: 2 \leq l_1, l_2 \leq m\}. \quad (2.6)$$

Set  $\tau(z', \delta) = \tau$  for a convenience and define

$$\begin{aligned} R_\delta(z') &= \{\zeta \in \mathbb{C}^n: |\zeta_k| \leq \tau, 1 \leq k \leq n-1, |\zeta_n| \leq \delta\}, \\ Q_\delta(z') &= \{\Phi_{z'}(\zeta): \zeta \in R_\delta(z')\}. \end{aligned} \quad (2.7)$$

Then by virtue of the definition of  $\tau(z', \delta)$ , it follows from (2.4) that  $r(z)$  is changed by no more than the given number  $\delta > 0$  in  $Q_\delta(z')$ .

For  $z' \in U \cap \Omega$  and  $\delta > 0$ , we define a biholomorphism (dilation map) defined by

$$D_{z'}^\delta(\zeta) = (\tau^{-1}\zeta_1, \dots, \tau^{-1}\zeta_{n-1}, \delta^{-1}\zeta_n) := (w_1, \dots, w_n),$$

and set

$$\rho_{z'}^\delta(w) = \delta^{-1}(\rho \circ (D_{z'}^\delta)^{-1}(w)).$$

Set  $b_j(\zeta) = (\partial\rho/\partial\zeta_n)^{-1}\partial\rho/\partial\zeta_j$ ,  $1 \leq j \leq n-1$ . In terms of dilated coordinates we set

$$\begin{aligned} L_j^\delta &= \tau(D_{z'}^\delta)_* L_j = \frac{\partial}{\partial w_j} - b_j \circ (D_{z'}^\delta)(w) \delta^{-1} \tau \frac{\partial}{\partial w_n}, \quad 1 \leq j \leq n-1, \\ L_n^\delta &= \delta(D_{z'}^\delta)_* L_n = \frac{\partial}{\partial w_n}. \end{aligned} \quad (2.8)$$

We need the following lemmas for a proof of Lemma 2.4.

**Lemma 2.2.** Let  $P_k(z, \bar{z}) = \sum_{i+j=k} d_{i,j} z^i \bar{z}^j$  be a homogeneous polynomial of order  $k$  in  $z$  and  $\bar{z}$  and suppose that  $|P_k(z, \bar{z})| \leq \epsilon$  for all  $z$  on the unit circle on  $\mathbb{C}^1$ . Then  $|d_{i,j}| \leq \epsilon$ .

**Proof.**  $P_k(z, \bar{z}) = \sum_{l+j=k} d_{l,j} e^{i(l-j)\theta}$  on the unit circle in  $\mathbb{C}^1$ . So

$$|d_{l,j}| = \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} P_k(z, \bar{z}) e^{i(l-j)\theta} d\theta \right| \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |P_k|_\infty d\theta \leq \epsilon. \quad \square$$

**Lemma 2.3** [3, Proposition 2.6]. Let  $P(z, \bar{z}) = \sum_{i+j \leq l} a_{ij} z^i \bar{z}^j$  be a polynomial of order  $l$  in  $\mathbb{C}^1$  with  $|a_{ij}| \leq 1$ . Suppose  $|P(z, \bar{z})| \leq \epsilon^2$  for all  $|z| \leq 1$  for some small number  $\epsilon > 0$ . Then  $|a_{ij}| \leq C_l \epsilon^\nu$ , where  $\nu = 1/l!$ , and where  $C_l$  depends only on  $l$ .

Now we want to show that the numbers  $b_{\alpha'\beta'}(z')$  in (2.4) are dominated by the quantities  $a_{jk}(z')$ . This will be shown by using the dilated coordinates, not using the vector fields  $L_j^\delta$ , and the proof depends heavily on the fact that the Levi form is comparable. Let  $0 < a \leq C^{-2}$  be a small constant to be determined where  $C$  is the constant satisfying (2.3).

**Lemma 2.4.** There are integers  $j, k$ ,  $j+k \leq m$ , such that  $|a_{jk}| \geq a^{(m!)^{2(n-2)nm}} \delta \tau^{-j-k}$ .

**Proof.** Suppose not. Then

$$|a_{jk}| < a^{(m!)^{2(n-2)nm}} \delta \tau^{-j-k} \quad \text{for all } j+k \leq m. \quad (2.9)$$

Assuming (2.9) we will show by induction that all the coefficients  $b_{\alpha'\beta'}$  in the expansion of  $\rho(\zeta)$  in (2.4) satisfy  $|b_{\alpha'\beta'}| \ll \delta \tau^{-|\alpha'+\beta'|}$ . Then this fact together the estimates in (2.9) contradict to the definition of  $\tau(z', \delta)$  in (2.6) and hence Lemma 2.4 follows.

Set  $\epsilon_l(k) = a^{(m!)^{2(n-l)nm-nk}}$ ,  $2 \leq l \leq n-1$ ,  $0 \leq k \leq m-1$ . By virtue of (2.6) it follows that  $\tau \ll \epsilon_2(0)$ , provided  $\delta > 0$ , is sufficiently small and we obtain the relations

$$\epsilon_2(k-1)\epsilon_2(l-1)^{(k-1)/l} \leq \epsilon_2(l-1)\epsilon_2(l-1)^{(l-1)/l}, \quad k \leq l. \quad (2.10)$$

In dilated coordinates, set  $\rho_2(w) = \rho_z^\delta(w_1, w_2, 0, \dots, 0, w_n)$  and write

$$\begin{aligned} \rho_2(w) = 2 \operatorname{Re} w_n + \sum_{\substack{j+k \leq m \\ j, k > 0}} a_{jk} \delta^{-1} \tau^{j+k} w_1^j \bar{w}_1^k \\ + \operatorname{Re} \sum_{1 \leq l_1+l_2 \leq m} P_{l_1, l_2}(\tau w_1) \delta^{-1} \tau^{l_1+l_2} w_2^{l_1} \bar{w}_2^{l_2} + \mathcal{O}(\tau), \end{aligned} \quad (2.11)$$

where  $P_{l_1, l_2}(\tau w_1)$  is a polynomial in  $\tau w_1$  and  $\tau \bar{w}_1$  of order less than or equal to  $m - l_1 - l_2$ . By virtue of the definition of  $L_i^\delta$ ,  $1 \leq i \leq n$ , in (2.8) and by (2.11) it follows that

$$\partial \bar{\partial} \rho_2(L_i, \bar{L}_j) = \frac{\partial^2 \rho_2}{\partial w_i \partial \bar{w}_j} + \mathcal{O}(\tau). \quad (2.12)$$

Set

$$P_{l_1, l_2}^\delta(w_1) = P_{l_1, l_2}(\tau w_1) \delta^{-1} \tau^{l_1+l_2}, \quad 1 \leq l_1 + l_2 \leq m.$$

Then the definition of  $\tau(z', \delta)$  shows that the coefficients of  $P_{l_1, l_2}^\delta(w_1)$  are bounded by one.

First we assume that  $l_1 + l_2 = 1$ . Since there is no pure (imaginary or real) terms in the expansion of  $\rho_2(w)$ , it follows that the order of  $P_{l_1, l_2}^\delta(w_1)$  is greater than or equal to one. Hence it follows from (2.2), (2.9), and (2.12) that

$$\begin{aligned} |\partial \bar{\partial} \rho_2(L_2^\delta, \bar{L}_1^\delta)(w_1, 0, \dots, 0, w_n)| &\lesssim |\partial \bar{\partial} \rho_2(L_1^\delta, \bar{L}_1^\delta)(w_1, 0, \dots, 0, w_n)| \\ &\leq a_{11} \delta^{-1} \tau^2 + \mathcal{O}(\tau) \leq \epsilon_2(0) + \mathcal{O}(\tau) \lesssim \epsilon_2(0). \end{aligned}$$

By (2.12) we then have

$$\left| \frac{\partial}{\partial \bar{w}_1} P_{1,0}^\delta(w_1) \right| \lesssim \epsilon_2(0) \quad \text{and} \quad \left| \frac{\partial}{\partial w_1} P_{0,1}^\delta(w_1) \right| \lesssim \epsilon_2(0).$$

Since there is no pure terms in the expansion of  $\rho_2(w)$ , these estimates show that

$$|P_{l_1, l_2}^\delta(w_1)| \lesssim \epsilon_2(0), \quad l_1 + l_2 = 1. \quad (2.13)$$

Then by Lemma 2.3, all the coefficients of  $P_{l_1, l_2}(w_1)$ ,  $l_1 + l_2 = 1$ , are bounded by  $\epsilon_2(0)^{1/2(m-1)!} \ll \epsilon_2(1)$ .

At the points where  $|w_2| = \epsilon_2(0)^{1/2}$ , it follows from (2.9)–(2.13) that

$$\partial \bar{\partial} \rho_2(L_1^\delta, \bar{L}_1^\delta) \lesssim \epsilon_2(0)^{1/2} \epsilon_2(1). \quad (2.14)$$

Now assume  $l_1 + l_2 = 2$ . If  $l_1 = l_2 = 1$ , it follows that

$$\partial \bar{\partial} \rho_2(L_2^\delta, \bar{L}_2^\delta)(w_1, 0, \dots, 0, w_n) \lesssim \partial \bar{\partial} \rho_2(L_1^\delta, \bar{L}_1^\delta)(w_1, 0, \dots, 0, w_n) \lesssim \epsilon_2(0).$$

From (2.12), we then have

$$|P_{1,1}^\delta(w_1)| \lesssim \epsilon_2(0), \quad (2.15)$$

and hence all the coefficients of  $P_{1,1}^\delta(w_1)$  are bounded by  $\epsilon_2(0)^{1/2(m-1)!} \ll \epsilon_2(1)$ .

Assume  $l_1 = 2$  (and hence  $l_2 = 0$ ). Then at the points where  $|w_2| = \epsilon_2(0)^{1/2}$ , it follows from (2.9)–(2.15) that

$$\partial \bar{\partial} \rho_2(L_2^\delta, \bar{L}_1^\delta) = \frac{\partial P_{2,0}^\delta(w_1)}{\partial \bar{w}_1} w_2 + \mathcal{O}(\epsilon_2(0)^{1/2} \epsilon_2(1)) + \mathcal{O}(\tau),$$

and hence we obtain from (2.2) and (2.14) that

$$\begin{aligned} \left| \frac{\partial}{\partial \bar{w}_1} P_{2,0}^\delta(w_1) \right| &= \epsilon_2(0)^{-1/2} \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} \partial \bar{\partial} \rho_2(L_2^\delta, \bar{L}_1^\delta) e^{-i\theta} d\theta \right| + \mathcal{O}(\epsilon_2(1)) \\ &\lesssim \epsilon_2(0)^{-1/2} \partial \bar{\partial} \rho_2(L_1^\delta, \bar{L}_1^\delta) + \mathcal{O}(\epsilon_2(1)) \lesssim \epsilon_2(1), \end{aligned}$$

because  $\tau \ll \epsilon_2(0)$  and  $\epsilon_2(0)^{1/2} \ll \epsilon_2(1)$ . Since  $P_{2,0}^\delta(w_1)w_2^2$  contains no pure terms, it follows that

$$|P_{2,0}^\delta(w_1)| \lesssim \epsilon_2(1). \quad (2.16)$$

When  $l_2 = 2$  and  $l_1 = 0$ , we get the estimates similar to (2.16) and hence we conclude from (2.15) and (2.16) that

$$|P_{l_1, l_2}^\delta(w_1)| \lesssim \epsilon_2(1) \quad \text{for } l_1 + l_2 = 2.$$

If we use Lemma 2.3 again we obtain that all the coefficients of  $P_{l_1, l_2}^\delta(w_1)$  are bounded by  $\epsilon_2(1)^{1/2(m-1)!} \ll \epsilon_2(2)$ ,  $l_1 + l_2 = 2$ .

Now let  $l \geq 2$  and assume by induction that

$$\begin{aligned} \left| \frac{\partial}{\partial \bar{w}_1} P_{l_1, 0}^\delta(w_1) \right| &\lesssim \epsilon_2(l_1 - 1), \quad \left| \frac{\partial}{\partial w_1} P_{0, l_2}^\delta(w_1) \right| \lesssim \epsilon_2(l_2 - 1), \quad \text{and} \\ |P_{l_1, l_2}^\delta(w_1)| &\lesssim \epsilon_2(l_1 + l_2 - 1) \end{aligned} \quad (2.17)$$

for all  $l_1 + l_2 \leq l$ , and assume that all the coefficients of  $P_{l_1, l_2}^\delta(w_1)$  are bounded by  $\epsilon_2(l_1 + l_2)$ .

Let  $l_1 + l_2 = l + 1 \geq 3$ . Combining (2.11), (2.12), and (2.17), one obtains that

$$\partial \bar{\partial} \rho_2(L_1^\delta, \bar{L}_1^\delta) \lesssim \sum_{k=1}^l \epsilon_2(k) |w_2|^k + \mathcal{O}(|w_2|^{l+1}) + \mathcal{O}(\epsilon_2(0)) + \mathcal{O}(\tau). \quad (2.18)$$

Since  $\partial \bar{\partial} \rho_2(L_2^\delta, \bar{L}_2^\delta)(w) \lesssim \partial \bar{\partial} \rho_2(L_1^\delta, \bar{L}_1^\delta)(w)$ , it follows from (2.18) that

$$\partial \bar{\partial} \rho_2(L_2^\delta, \bar{L}_2^\delta)(w) \lesssim \sum_{k=1}^l \epsilon_2(k) |w_2|^k + \mathcal{O}(|w_2|^{l+1}) + \mathcal{O}(\epsilon_2(0)) + \mathcal{O}(\tau). \quad (2.19)$$

Combining (2.12), (2.17), (2.19), and by induction hypothesis, one obtains that

$$\begin{aligned} \left| \sum_{\substack{l_1+l_2=l+1 \\ l_1, l_2 \geq 1}} P_{l_1, l_2}^\delta(w_1) w_2^{l_1-1} \bar{w}_2^{l_2-1} \right| &\lesssim |\partial \bar{\partial} \rho_2(L_2^\delta, \bar{L}_2^\delta)(w)| + \sum_{k=1}^{l-1} \epsilon_2(k-1) |w_2|^{k-1} \\ &\quad + \mathcal{O}(|w_2|^l) + \mathcal{O}(\epsilon_2(0)) \\ &\lesssim \epsilon_2(l-1) + \mathcal{O}(|w_2|^l) + \epsilon_2(1) |w_2|. \end{aligned} \quad (2.20)$$

We integrate the quantity inside of  $|\cdot|$  in the left side of (2.20) along  $|w_2| = \epsilon_2(l-1)^{1/l}$ . Then we obtain, from Lemma 2.2, that

$$\begin{aligned} |P_{l_1, l_2}^\delta(w_1)| &\lesssim \epsilon_2(l-1)^{-1+1/l} (\epsilon_2(l-1) + \epsilon_2(1) \epsilon_2(l-1)^{1/l}) \\ &\lesssim \epsilon_2(l-1)^{1/l}, \quad l_1, l_2 \geq 1, \quad l_1 + l_2 = l+1, \end{aligned} \quad (2.21)$$

because  $l_1 + l_2 - 2 = l - 1$ . Hence all the coefficients of  $P_{l_1, l_2}(w_1)$ ,  $l_1, l_2 \geq 1$ ,  $l_1 + l_2 = l + 1$ , are bounded by  $(\epsilon_2(l-1)^{1/l})^{1/(2(m-l))} \ll \epsilon_2(l)$ .

Now assume that  $l_1 = l + 1$  (and so  $l_2 = 0$ ). Then by virtue of (2.11), (2.12), (2.17), and (2.21), we can write

$$\begin{aligned} \partial \bar{\partial} \rho_2(L_2^\delta, \bar{L}_1^\delta) &= \frac{\partial P_{l+1, 0}^\delta(w_1)}{\partial \bar{w}_1} w_2^l + \mathcal{O}\left(\sum_{k=1}^l \epsilon_2(k-1) |w_2|^{k-1}\right) \\ &\quad + \epsilon_2(l) |w_2|^l + \mathcal{O}(|w_2|^{l+1}) + \mathcal{O}(\tau). \end{aligned} \quad (2.22)$$

Combining (2.10), (2.18), (2.22), and integrating along  $|w_2| = \epsilon_2(l-1)^{1/l}$ , one obtains that

$$\begin{aligned} &\left| \frac{\partial}{\partial \bar{w}_1} P_{l+1, 0}^\delta(w_1) \right| \\ &= \epsilon_2(l-1)^{-1} \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} \partial \bar{\partial} \rho_2(L_2^\delta, \bar{L}_1^\delta)(w_1, \epsilon_2(l-1)^{1/l} e^{i\theta}, 0, \dots, 0, w_n) e^{-il\theta} d\theta \right| \\ &\quad + \mathcal{O}\left(\epsilon_2(l-1)^{-1} \left( \sum_{k=1}^l \epsilon_2(k-1) \epsilon_2(l-1)^{(k-1)/l} \right)\right) \\ &\quad + \epsilon_2(l-1)^{-1} \epsilon_2(l) \epsilon_2(l-1) + \mathcal{O}(\epsilon_2(l-1)^{-1} \epsilon_2(l-1)^{(l+1)/l}) \\ &\lesssim \epsilon_2(l-1)^{-1} \partial \bar{\partial} \rho_2(L_1^\delta, \bar{L}_1^\delta) + \mathcal{O}(\epsilon_2(l)) \lesssim \epsilon_2(l), \end{aligned}$$

because  $\epsilon_2(l-1)^{1/l} \ll \epsilon_2(l)$ . Similarly, if  $l_2 = l + 1$  (and hence  $l_1 = 0$ ), we obtain that

$$\left| \frac{\partial}{\partial w_1} P_{0, l+1}^\delta(w_1) \right| \lesssim \epsilon_2(l).$$



If we apply Lemma 2.3 again, we obtain (2.18) with  $l_1, l_2$  replaced by  $l_1 + l_2 = l + 1$ , and hence all the coefficients of  $P_{l_1, l_2}^\delta(w_1)$  are bounded by  $\epsilon_2(l_1 + l_2)$  for  $l_1 + l_2 \leq l + 1$ . We proceed this process up to  $l_1 + l_2 = m$ . Then we obtain that all the coefficients of  $P_{l_1, l_2}(w_1)$ ,  $1 \leq l_1 + l_2 \leq m$ , are bounded by  $\epsilon_2(m) = \epsilon_3(0)$ .

Now we want to estimate the coefficients in the directions other than  $w_1$  and  $w_2$ . Set  $w^{(l)} = (w_1, \dots, w_l, 0, \dots, 0, w_n)$ ,  $2 \leq l \leq n - 1$ , set  $\rho_{k+1}(w) = \rho_{z'}^\delta(w^{(k+1)})$ , and write

$$\begin{aligned} \rho_{k+1}(w) &= 2 \operatorname{Re} w_n + \sum_{j, k \geq 1} a_{jk} \delta \tau^{j+k} w_1^j \bar{w}_1^k \\ &\quad + \operatorname{Re} \sum_{0 \leq l_1 + l_2 \leq m} P_{l_1, l_2}(w^{(k)}) w_{k+1}^{l_1} \bar{w}_{k+1}^{l_2} + \mathcal{O}(\tau). \end{aligned}$$

Assume, by induction (on  $k$ ), that all the coefficients of  $P_{0,0}(w^{(k)})$  are bounded by  $\epsilon_{k+1}(0)$ . (This holds for  $k = 1$  by (2.9) and we have just proved that  $P_{0,0}(w^{(2)})$  are bounded by  $\epsilon_3(0)$ .) Then as in the case of  $k = 1$ , we use the relations

$$\begin{aligned} |\partial \bar{\partial} \rho_{k+1}(L_{k+1}, \bar{L}_j)| &\lesssim \partial \bar{\partial} \rho_{k+1}(L_1, \bar{L}_1), \quad 1 \leq j \leq k + 1, \\ \partial \bar{\partial} \rho_{k+1}(L_i, \bar{L}_j) &= \frac{\partial^2 \rho_{k+1}}{\partial w_i \partial \bar{w}_j} + \mathcal{O}(\tau), \quad 1 \leq i, j \leq k + 1, \end{aligned}$$

and replacing  $\epsilon_2(l)$  by  $\epsilon_{k+1}(l)$  we get, for  $l_1 + l_2 = l$ , that

$$\begin{aligned} \left| \frac{\partial}{\partial \bar{w}_j} P_{l_1, 0}(w^{(k)}) \right| &\lesssim \epsilon_{k+1}(l - 1), \quad \text{if } l_2 = 0, \\ \left| \frac{\partial}{\partial w_j} P_{0, l_2}(w^{(k)}) \right| &\lesssim \epsilon_{k+1}(l - 1), \quad \text{if } l_1 = 0, \end{aligned}$$

and

$$|P_{l_1, l_2}(w^{(k)})| \lesssim \epsilon_{k+1}(l - 1)$$

for all  $1 \leq j \leq k$ . Again if we apply Lemmas 2.2 and 2.3 successively, we will get that all the coefficients of  $P_{l_1, l_2}(w^{(k)})$  are bounded by  $\epsilon_{k+1}(l)$ .

By induction on  $l$ , and then on  $k$ , we conclude that the coefficients  $b_{\alpha' \beta'}$  in (2.4) satisfy

$$|b_{\alpha' \beta'}(z')| \lesssim \epsilon_{n-1}(m) \delta \tau^{-|\alpha' + \beta'|} = a^{(m!)nm} \delta \tau^{-|\alpha' + \beta'|} \ll \delta \tau^{-|\alpha' + \beta'|}, \quad (2.23)$$

provided  $a > 0$  is sufficiently small. By (2.9) and by virtue of the definition of  $\tau(z', \delta)$ , this cannot happen and hence we have proved Lemma 2.4.  $\square$

**Remark 2.5.** (1) Note that all the inequalities in the proof of Lemma 2.4 depend only on the dimension  $n$  and the type  $m$  of the point  $z_0 \in b\Omega$ . Therefore there is a sufficiently small  $a_0 = a_0(m, n) > 0$  such that Lemma 2.4 is true.

(2) Let  $a_0 > 0$  be a small constant that Lemma 2.4 holds, and set  $b_0 = a_0^{(m!)^{2(n-2)nm}}$ . Then Lemma 2.4 shows that there are  $j_0, k_0 \leq m$  such that

$$|a_{j_0 k_0}| \geq b_0 \delta \tau^{-j_0 - k_0}. \quad (2.24)$$

### 3. Estimates on the Bergman kernel function

In this section we prove the main theorem (Theorem 1.1) of this paper. For each  $z' \in U$  and  $\delta > 0$ , set

$$T(z', \delta) = \min\{l_1, A_{l_1}(z') \geq b_0 \delta \tau^{-l_1}\}, \quad (3.1)$$

where  $b_0 > 0$  is the number in (2.24). Then this  $T(z', \delta)$  represents the local geometry of the domain corresponding to the type condition [2–4]. Moreover, if we use (2.23), (2.24), and the Taylor's theorem argument together with the definition of  $\tau(z', \delta)$ , it follows, as in [2–4], that there is a small constant  $d > 0$  satisfying

$$|\mathcal{L}_{j_0, k_0} \partial \bar{\partial} \rho(\zeta)| \approx \delta \tau^{-j_0 - k_0}, \quad |\mathcal{L}_{j, k} \partial \bar{\partial} \rho(\zeta)| \lesssim \delta \tau^{-j - k} \quad (3.2)$$

for  $\zeta \in R_{d\delta}(z')$ .

For  $\epsilon > 0$ , we let  $\Omega_\epsilon = \{z; r(z) < \epsilon\}$  and set

$$S(\epsilon) = \{z; -\epsilon < r(z) < \epsilon\}.$$

Note that the properties in (2.24), (3.1), and (3.2) are the key ingredients in the construction of the plurisubharmonic functions with maximal Hessian in a thin strip of  $b\Omega$  [2]. Therefore if we follow the Catlin's construction of weight functions in [2], we then have the following theorem.

**Theorem 3.1.** *For all small  $\delta > 0$ , there is a plurisubharmonic function  $\lambda_\delta \in C^\infty(\Omega_\delta)$  with the following properties:*

- (i)  $|\lambda_\delta(z)| \leq 1, z \in U \cap \Omega_\delta$ ;
- (ii) For all  $L = \sum_{j=1}^n b_j L_j$  at  $z \in U \cap S(\delta)$ ,

$$\partial \bar{\partial} \lambda_\delta(z)(L, \bar{L}) \approx \tau^{-2} \sum_{k=1}^{n-1} |b_k|^2 + \delta^{-2} |b_n|^2;$$

- (iii) If  $\Phi_{z'}$  is the map associated with a given  $z' \in U \cap S(\delta)$ , then for all  $\zeta \in R_\delta(z')$  with  $|\rho(\zeta)| < \delta$ ,

$$|\partial^\alpha \bar{\partial}^\beta (\lambda_\delta \circ \Phi_{z'})(\zeta)| \lesssim C_{\alpha, \beta} \delta^{-\alpha_n - \beta_n} \tau^{-|\alpha' + \beta'|}.$$

**Remark 3.2.** Since  $\tau(z', \delta) \lesssim \delta^{1/m}$ , the existence of the family  $\{\lambda_\delta\}_{\delta>0}$  with maximal Hessian as in Theorem 3.1 implies that the sharp subelliptic estimates of order  $1/m$  holds for the  $\bar{\partial}$ -Neumann problem for the  $(0, 1)$ -forms by Catlin's theorem [1].

We are now ready to prove Theorem 1.1.

**Proof of Theorem 1.1.** Let  $z \in U$  with  $r(z) = -d\delta/2$  and  $\pi(z) = z' \in b\Omega$  be the projection of  $z$  onto  $b\Omega$ . Here  $d > 0$  is the number satisfying (3.2) in  $R_{d\delta}(z')$ . Set  $w = (0, \dots, 0, -d\delta/2)$ , and let  $\Phi_{z'}$  be the biholomorphism defining special coordinates

in Proposition 2.1. Then by virtue of (2.4), (2.6), and (2.7), it follows that there is a constant  $0 < c \leq d$  such that the polydisc

$$B = \{\zeta: |\zeta_k| < c\tau(z', \delta), k = 1, \dots, n-1, |\zeta_n + d\delta/2| < c\delta\}$$

lies in  $\Omega_{z'} := \Phi_{z'}^{-1}(\Omega)$ . Moreover, if we set  $\phi_\delta(\zeta) = \lambda_\delta \circ \Phi_{z'}(\zeta)$ , where  $\lambda_\delta$  is the plurisubharmonic function in Theorem 3.1, then  $\phi_\delta$  will satisfy all the properties of Theorem 3.1 in  $\zeta = (\Phi_{z'})^{-1}(z)$  coordinates, especially,

$$\partial\bar{\partial}\phi_\delta(\zeta)(L, \bar{L}) \approx \tau(z', \delta)^{-2} \sum_{k=1}^{n-1} |b_k|^2 + \delta^{-2} |b_n|^2$$

for all  $L = \sum_{j=1}^n b_j L_j$  at  $\zeta \in B$ .

Then by Theorem 6.1 in [2], which uses the existence of these polydiscs  $B$  and the functions  $\phi_\delta$  together with the standard weighted  $L^2$  estimates for  $\bar{\partial}$  [11], it follows that

$$K_{\Omega_{z'}}(w, \bar{w}) \approx \delta^{-2} \tau(z', \delta)^{-2(n-1)}. \quad (3.3)$$

Since the Jacobian of  $\Phi_{z'}$  at  $w$  satisfies  $|J_w(\Phi_{z'})| \approx 1$ , the transformation identity of the Bergman kernel function implies that

$$K_\Omega(z, \bar{z}) = |J_w(\Phi_{z'})|^{-2} K_{\Omega_{z'}}(w, \bar{w}) \approx \delta^{-2} \tau(z', \delta)^{-2(n-1)}.$$

Also, from (1.3), (1.4), (3.2), and by functoriality, it follows that

$$\tau(z, \delta)^{-2} \approx \sum_{l=2}^m |C_l(z)|^{2/l} |r(z)|^{-2/l} \quad (3.4)$$

for  $z \in Q_\delta(z')$  because  $|r(z)| \approx \delta$ . Therefore we conclude, from (3.3) and (3.4), that

$$K_\Omega(z, \bar{z}) \approx |r(z)|^{-2} \tau(z, \delta)^{-2(n-1)} \approx |r(z)|^{-2} \sum_{l=2}^m C_l(z)^{2(n-1)/l} |r(z)|^{-2(n-1)/l}.$$

The proof of Theorem 1.1 is now completed.  $\square$

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